Planar Quantum Mechanics: an Intriguing Supersymmetric Example

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Abstract

After setting up a Hamiltonian formulation of planar (matrix) quantum mechanics, we illustrate its effectiveness in a non-trivial supersymmetric example. The numerical and analytical study of two sectors of the model, as a function of 't Hooft's coupling λ, reveals both a phase transition at $\lambda = 1$ (disappearence of the mass gap and discontinuous jump in Witten's index) and a new form of strong-weak duality for $\lambda \to 1/\lambda$.

1 Introduction

More than thirty years after its introduction [1] the large-N limit of four-dimensional QCD remains elusive. It is widely believed that such a limit should capture the most interesting non-perturbative properties of QCD, such as confinement and spontaneous chiral symetry breaking, while neglecting others (e.g. resonance widths and the $U(1)$ anomaly). It is also believed that, in 't Hooft's limit, QCD should lend itself to an effective description in terms of a string theory (see e.g. [2] for recent developments of the subject) or, perhaps, of a gravitational dual similar to the one enjoyed by $\mathcal{N} = 4$ super-Yang–Mills (SYM) theory through the AdS-CFT correspondence [3]. Indeed, the large-N classification of diagrams according to topology closely resembles the loop expansion of string theory in terms of surfaces of increasing genus.

One should keep in mind, however, that the connection between large-N and graph-topology is only proven order-by-order in perturbation theory. Whether the true large-N limit (defined as solving exactly the theory at finite N and *then* taking N to infinity) does actually coincide with the non-perturbative solution of a suitably defined "lowest genus" theory remains to be proven case by case. Already the Gross-Witten model [4] and the work of Marinari and Parisi [5] have taught us that the $N \to \infty$ limit may not commute with other limits, such as the full resummation of the strong-coupling expansion or approaching first a phase transition. In QCD itself, the assumption (now supported by lattice calculations [6]) that the topological susceptibility depends on whether the large-N limit is taken before or after the chiral limit provides a solution of the $U(1)$ problem [7].

In this letter we shall consider a class of planar matrix models in a (possibly new) Hamiltonian formulation. (Large-N literature being very vast, we shall refer to the nice reprint collection [8] for the classic papers on the subject). We shall then solve a particular supersymmetric case (both numerically and analytically) in a planar (or better lowest-genus) approximation and point out several amusing features of the solution, including a phase transition and a non-trivial strong-weak duality in 't Hooft's coupling λ.

One motivation for this work was to prepare the ground for checking, in a simpler context, a recently claimed planar equivalence [9] between a supersymmetric "parent theory" and its non-supersymmetric "daughter" in a particular subsector. Our method (or at least its numerical part) should apply without major modifications to the latter theory, and therefore such a check should be forthcoming. Another motivation came from the recent studies of the supersymmetric Yang-Mills quantum mechanics in various dimensions [10]. Although done at present mainly for SU(2) [11] (and partly for SU(3) and SU(4) [12]) gauge groups, the goal of these works is to extrapolate eventually towards large N whereby making contact with M-theory [13]. Our results here offer the prospect of a substantial shortcut for the whole program.

The rest of this letter (see [14] for more detailed account) is organized as follows: we first define a general class of planar quantum mechanics (PQM) models and specify some conditions for our method to be applicable. We then focus on the supersymmetric case and, eventually, on a particular example for which spectrum and main features can be worked out both numerically and analytically.

2 Hamiltonian formulation of PQM

Our formulation of planar quantum mechanics (PQM) is best done directly in a standard Hamiltonian framework (see also [15]). Let us start by defining a Hilbert space (with states that span it) and operators (acting on it). The operators will be $N \times N$ destruction and creation matrices:

$$^{(k)}M_i^j, \quad {}^{(k)}M_i^{j\dagger}, \, i,j = 1, \ldots N \quad ; \quad k = 1, \ldots N_f \,, \tag{1}$$

where i, j represent "colour" indices while k represents a generalized "flavour" index. The latter can be used, in a QFT generalization, as a label for momenta, polarizations etc. In this paper it will only serve the purpose of distinguishing bosonic and fermionic degrees of freedom. The above operators are assumed to obey standard (anti) commutation relations. In familiar notations:

$$\left[{}^{(k)}M_i^j, \, {}^{(k')}M_l^{m\dagger} \right\} = \delta_i^m \delta_l^j \delta_{k,k'} \,. \tag{2}$$

The Hilbert space is constructed out of the usual Fock vacuum (annihilated by all $^{(k)}M_i^j$) by acting on it with a *single-trace* string of creation operators. This is the first essential difference between general and planar QM. It is of course mimiking the colour structure of the states that propagate in genus-zero diagrams, a small subset of all states that are singlets of a $U(N)$ group acting as:

$$M \to UMU^\dagger \,. \tag{3}$$

This is of course an enormous simplification: if, for instance, the index k takes a single (bosonic) value, the states spanning the Hilbert space are just labelled by a single integer, n, corresponding to the number of $M^\dagger$s in the trace (see below).

The Hamiltonians we shall consider are also single-trace operators and therefore singlets of $U(N)$, but, of course, they will include both creation and destruction operators. In order to avoid producing unwanted vacuum diagrams we shall impose that the Hamiltonian H annihilates the Fock vacuum. This typically implies (though it is not implied by) normal ordering of the operators appearing in H (not to be confused, of course, with the order in the colour trace).

The Hamiltonian will be a sum of such single-trace operators and will contain a factor g^{n-2} for a term containing n operators. Schematically:

$$H = \sum_n c_n g^{n-2} Tr(M^n) \,, \tag{4}$$

where M^n stands for a product of Ms and $M^\dagger$s with a total of n factors. The ('t Hooft) limit to be consider is, as usual, $N \to \infty$ with the 't Hooft coupling $\lambda \equiv g^2 N$ kept fixed.

When such a Hamiltonian acts on a generic single-trace state it will not give, generically, another single-trace state. However, whenever it does not, one gets subleading terms in the large-N limit. If we discard such terms we have a closed system and the

matrix elements of the Hamiltonian turn out to be functions of λ alone: we simply have a well-defined Hamiltonian to diagonalize in the single-trace Hilbert space.

The final ingredient of our approach is to introduce a cut-off $B \equiv n_{max}$ in the occupation number thus reducing the problem to one that can be managed numerically [10] . Eventually, by increasing B, we can check whether the lowest eigenstates and eigenvalues converge to some finite limit. As we shall see, this will be the case in a simple toy model where the method gives very interesting indications of the dependence of the spectrum from λ. In turn, the numerical results will suggest properties that we shall be able to derive analytically.

3   A class of supersymmetric matrix models

We will now specialize to the case in which there is just one bosonic and one fermionic matrix, denoted, respectively, by a and f (plus their Hermitian conjugates).

The class of supersymmetric models that we consider are a straightforward matrix generalization [5] of Witten's supersymmetric quantum mechanics (SQM)[16]. We will assume that the supersymmetric charges Q and $Q^\dagger$ are single-trace operators that are linear in the fermionic matrices f and $f^\dagger$. Thus:

$$Q = Tr(A^\dagger f) , \quad Q^\dagger = Tr(A f^\dagger) , \tag{5}$$

where $A = A(a, a^\dagger)$ represents some function of the bosonic matrices. We also demand that Q and $Q^\dagger$ are nihlpotent. This gives:

$$[A^i_j, A^k_l] = 0 , \quad \text{for all } i, j, k, l . \tag{6}$$

In our explicit example we will satisfy this condition trivially by making A $(A^\dagger)$ depend only on a $(a^\dagger)$. We then obtain the supersymmetric Hamiltonian as:

$$H = \{Q^\dagger, Q\} = (f^\dagger)^i_j \ f^k_l [A^j_i, A^{\dagger l}_k] + A^{\dagger i}_j \ A^j_i . \tag{7}$$

By construction, H commutes with Q and $Q^\dagger$. It also commutes with fermion number, $F = Tr(f^\dagger \ f)$. In order to get rid of disconnected diagrams we need the condition:

$$A^i_j |0> = 0 , \quad \text{for all } i, j , \tag{8}$$

which is again trivially satisfied if A depends only on a, something that guarantees that also Q and $Q^\dagger$ annihilate the trivial (empty) Fock state. Thus, by construction, our model has (at least) one zero-energy state and does not break supersymmetry.

The spectrum of the theory should then consist of a zero-energy sector (providing a certain value of Witten's index) and degenerate massive supermultiplets. It is easy to show[1] that, barring unexpected extra symmetries, these supermultiplets should consist of just one boson and one fermion. Technically, this is a consequence of the fact that the

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[1]One of us (GV) wishes to thank A. Veinshtein for a useful discussion on this issue.

only other operator in the SUSY algebra (besides Q, $Q^\dagger$ and H), $C \equiv [Q^\dagger, Q]$, satisfies the equation $C^2 = H^2$. Hence, eigenstates can be classified according to whether they carry $C = \pm H = \pm E$ (in amusing analogy with BPS states). Furthermore, the algebra implies that states with positive (negative) C are annihilated by $Q^\dagger$ (Q), while they are transformed in a state with opposite C by the other supersymmetric charge. All non-zero-energy levels must therefore consist of two states with opposite C-parity. For the $F = 0$ ($F = 1$) sector C is negative (positive) for *all* the states but this fails to be the case for higher values of F (see Section 6).

4   A specific model and its numerical analysis

We now specialize further our model by taking:

$$Q = Tr[fa^\dagger(1 + ga^\dagger)] = Tr[fA^\dagger], \quad Q^\dagger = Tr[f^\dagger(1 + ga)a] = Tr[f^\dagger A], \tag{9}$$

and therefore

$$H = H_B + H_F\,; \tag{10}$$

$$H_B = Tr[a^\dagger a + g(a^{\dagger^2} a + a^\dagger a^2) + g^2 a^{\dagger^2} a^2]\,; \tag{11}$$

$$
\begin{aligned}
H_F = Tr[&f^\dagger f + g(f^\dagger f(a^\dagger + a) + f^\dagger(a^\dagger + a)f) \\
&+ g^2(f^\dagger afa^\dagger + f^\dagger aa^\dagger f + f^\dagger fa^\dagger a + f^\dagger a^\dagger fa)]\,.
\end{aligned}
\tag{12}
$$

In most of this paper we shall limit our attention to the $F = 0$ and $F = 1$ sectors. Some discussion of our expectations for the $F \geq 2$ will be given at the end of the paper but a detailed analysis is postponed to further work.

As already anticipated, the planar states in the $F = 0$ sector are simply labeled by the integer n corresponding to the number of $a^\dagger$s in the trace. We shall denote the normalized state with n bosonic quanta by $|0, n\rangle$. Similarly, the generic (single-trace) $F = 1$ normalized state will be denoted by $|1, n-1\rangle$: the corresponding creation operator contains one fermionic and $n - 1$ bosonic operators. In the free theory ($g = 0$) there is a single zero-energy bosonic state, $|0, 0\rangle$, while $|0, n\rangle$ and $|1, n - 1\rangle$ form a supermultiplet.

Working out the matrix elements of the Hamiltonian is straightforward although tedious (in particular the normalization factors have to be kept accurately). Keeping only the leading terms as $N \to \infty$ we find that the final result for the matrix elements of H depend only on $\lambda \equiv g^2 N$, and are given by:

$$\langle 0, n|H|0, n\rangle = (1 + \lambda(1 - \delta_{n1}))\, n\,, \tag{13}$$

$$\langle 0, n + 1|H|0, n\rangle = \langle 0, n|H|0, n+1\rangle = \sqrt{\lambda}\sqrt{n(n+1)}\,, \tag{14}$$

$$\langle 1, n|H|1, n\rangle = (n + 1)(1 + \lambda) + \lambda\,, \tag{15}$$

$$\langle 1, n + 1|H|1, n\rangle = \langle 1, n|H|1, n+1\rangle = \sqrt{\lambda}(2 + n)\,. \tag{16}$$

After introducing a cutoff $B \equiv n_{max}$ we can diagonalize the Hamiltonians in the two sectors and compute the spectra. Eigenvalues with $E << B$ converge rapidly to some finite values, except near $\lambda = 1$, where some critical slowdown of the calculation emerges. Fig. 1 gives the lowest fermionic and bosonic eigenvalues as functions of λ. Apart from the trivial bosonic ground state we observe that:

- There is excellent boson-fermion degeneracy if we stay away from $\lambda = 1$;

- The cutoff B explicitly breaks supersymmetry, which we expect to recover only at $B = \infty$;

- The breaking of SUSY allows the supermultiplets to split near $\lambda = 1$. More amusingly, above $\lambda = 1$, the supermultiplets form once more, but with new partners. The $|0, E_1\rangle$ state remains unpaired (with zero energy), while $|0, E_{n+2}\rangle$ pairs with $|1, E_n\rangle$ ($n = 0, 1, \ldots$) rather than with the small-coupling partner $|1, E_{n+1}\rangle$;

- Eigenvalues tend to collapse to zero at $\lambda = 1$ as the cutoff is increased.

- Some kind of symmetry between strong and weak 't Hooft coupling appears.

Obviously, the behaviour near $\lambda = 1$ is strongly suggestive of a phase transition (if the cutoff is removed). A rather shocking way of showing this is to plot the Witten index and partition function (restricted to the two sectors we have considered):

$$W(\beta, \lambda) \equiv Tr\left((-1)^F e^{-\beta H}\right) \quad , \quad Z(\beta, \lambda) \equiv Tr\left(e^{-\beta H}\right) \qquad (17)$$

The results are shown in Fig. 2. The sudden jump by one unit in $W(\beta, \lambda)$ around $\lambda = 1$ is quite spectacular. The standard, thermal partition function shows even more dramatic singularity at $\lambda = 1$. As expected, the large cutoff and large β limits do not commute. Our numerical results suggest that the large cutoff limit at fixed β reveals a singularity of Z at $\lambda = 1$ which tends to a step-function (i.e. as for W) plus a kind of "delta function" as $\beta \to \infty$.

In order to understand better these numerical results and what they mean at infinite cutoff, we now resort to some analytic methods.

5   Analytic solution

Let us introduce new "composite" creation and annihilation operators for single trace states:

$$a^\dagger_{\ n} (a_n) \text{ creates (annihilates) } |0, n\rangle \ ; \ f^\dagger_n, (f_n) \text{ creates (annihilates) } |1, n - 1\rangle , \qquad (18)$$

that (anti)commute as usual. Introducing for convenience $b \equiv \sqrt{\lambda}$, it is easy to show that:

$$H^{(F=0)} = a^\dagger_{\ 1} a_1 + \sum_{n=2}^{\infty} n(1 + b^2) a^\dagger_{\ n} a_n + \left(\sum_{n=1}^{\infty} b\sqrt{n(n+1)} a^\dagger_{\ n} a_{n+1} + h.c.\right) \qquad (19)$$
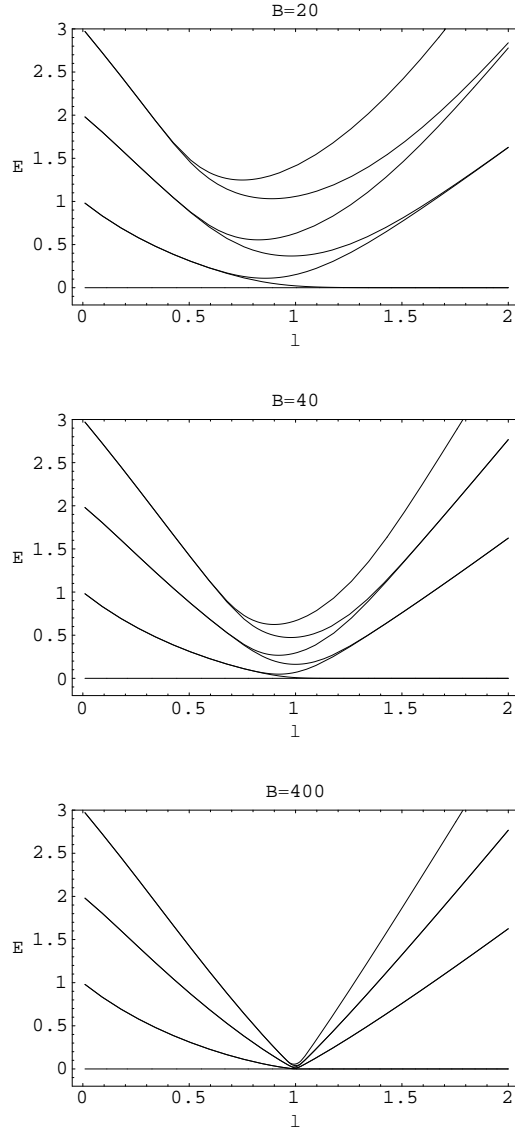
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Figure 1: Lowest bosonic and fermionic levels as functions of λ for different cutoffs

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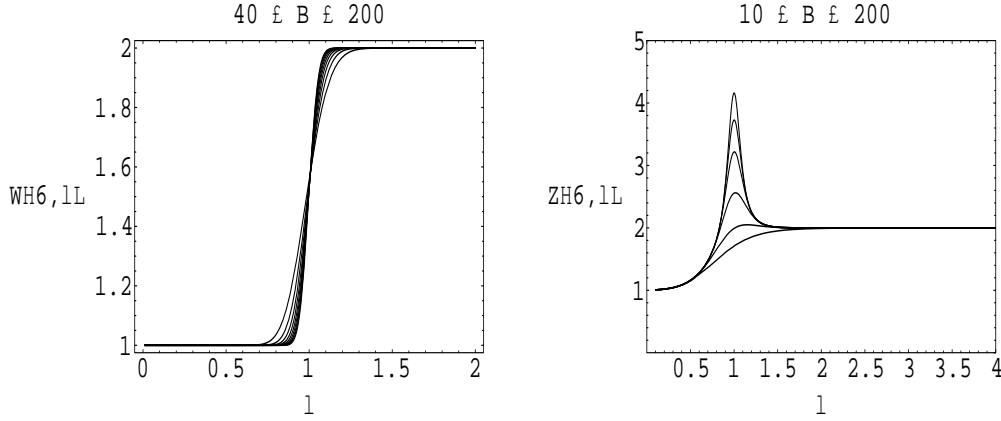Figure 2: λ dependence of the Witten index and the partition functiom, at $\beta = 6$ for different cutoffs.

and

$$H^{(F=1)} = \sum_{n=1}^{\infty}[n + (n+1)b^2]f_n^{\dagger}f_n + (\sum_{n=1}^{\infty} b(n+1)f_n^{\dagger}f_{n+1} + h.c.) \tag{20}$$

We can also construct the SUSY charges as:

$$Q = a^{\dagger}{}_1 f_1 + \sum_{n=1}^{\infty} \sqrt{n+1}a^{\dagger}{}_{n+1}(f_{n+1} + bf_n), \tag{21}$$

(and similarly for $Q^{\dagger}$) and check that $\{Q, Q^{\dagger}\} = H$.

In the $F = 0$ sector, besides the trivial vacuum, $|0\rangle_1$, one can formally construct a second state annihilated by H:

$$|0\rangle_2 = \sum_{n=1}^{\infty} \left(\frac{-1}{b}\right)^n \frac{a^{\dagger}{}_n}{\sqrt{n}}|0\rangle_1. \tag{22}$$

Clearly its norm is only finite at $b > 1$ explaining why there is no such a zero-energy state below $b = 1$. Using the formula given above for $Q^{\dagger}$ one can also check that $Q^{\dagger}|0\rangle_2 = 0$.

We now come to the derivation of a new sort of strong-weak-λ duality which surprisingly exists in this model. Using (20) for both b and $1/b$ we find immediately:

$$bH^{(F=1)}(1/b) = \frac{1}{b}H^{(F=1)}(b) + (1/b - b), \tag{23}$$

since $\sum_n f_n^{\dagger}f_n = 1$ in this sector. Because of SUSY it must also work in the $F = 0$ sector. In terms of eigenenergies, the duality relations read:

$$b\left(E_n^{(F=1)}(1/b) + 1\right) = \frac{1}{b}\left(E_n^{(F=1)}(b) + 1\right) ; \ b\left(E_n^{(F=0)}(1/b) + 1\right) = \frac{1}{b}\left(E_{n+1}^{(F=0)}(b) + 1\right). \tag{24}$$

b	1/2	2	3/4	4/3
n B				
10	3.18818	3.19736	1.68608	1.91640
1 20	3.18808	3.18808	1.67497	1.68238
30	3.18808	3.18808	1.67488	1.67499
exact	3.18807663		1.67488116	
20	6.45527	6.45637	3.09853	3.45017
3 40	6.45524	6.45524	2.97193	2.97543
60	6.45524	6.45524	2.97177	2.97177
exact	6.45523985		2.97177086	
30	9.49504	9.49513	4.46674	4.84625
5 50	9.49503	9.49503	4.19908	4.21714
70	9.49503	9.49503	4.19672	4.19675
exact	9.49503451		4.19671404	

Table 1: The cutoff dependence of three eigenenergies at two pairs of b values related by the duality described in the text. Exact results (c.f. below) are identical for dual partners.

Notice that, due to the existence of the second vacuum for $b > 1$, states, in the $F = 0$ sector, whose energies are related by duality, *do not* have the same n. These duality relations are nicely satisfied by our numerical eigenvalues computed at large cutoff (see Table 1).

A consequence of duality is that, for all levels, the left and right derivatives of E at $\lambda = 1$ ($E'_<, E'_>$) should satisfy :

$$E'_> + E'_< = 1/2 \tag{25}$$

a relation that has also been checked numerically.

We finally turn to an analytic determination of the massive spectrum. To this purpose it is convenient to rewrite H in the $F = 0$ subspace as:

$$H = \sum_{n=1}^{\infty} B_n^\dagger B_n \ , \quad B_n = \sqrt{n} a_n + b\sqrt{n+1} a_{n+1} \ ,$$

and to introduce new states:

$$|B_n\rangle \equiv B_n^\dagger |0\rangle = \sqrt{n}|n\rangle + b\sqrt{n+1}|n+1\rangle \ . \tag{26}$$

These states are not orthonormal, nevertheless they form a complete set and this suffices for our construction. The action of a "reduced" Hamiltonian, $\bar{H} \equiv (H - b^2)/b$, on the $|B_n >$ states is very simple:

$$\bar{H}|B_n\rangle = n|B_{n-1}\rangle + n\left(b + \frac{1}{b}\right)|B_n\rangle + (n+1)|B_{n+1}\rangle, \quad n = 2, \ldots . \tag{27}$$

with two exceptions at $n = 0, 1$ for which:

$$\bar{H}|B_0\rangle = -b|B_0\rangle + |B_1\rangle, \qquad \bar{H}|B_1\rangle = \left(b + \frac{1}{b}\right)|B_1\rangle + 2|B_2\rangle.$$

The simplicity of (27) allows us to map the eigenproblem of $\bar{H}$ into a simple differential equation. Let us expand any generic eigenstate of $\bar{H}$ into the $|B_n\rangle$ basis and associate with it a function of one variable x:

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |B_n\rangle , \qquad \leftrightarrow \qquad f(x) = \sum_{n=0}^{\infty} c_n x^n , \tag{28}$$

which is in fact a generating function for the $\{c_n\}$ coefficients.

It is then easy to see that the eigenequation for $|\psi\rangle$ maps into the following first order differential equation for $f(x)$:

$$w(x)f'(x) + xf(x) - bf(0) - f'(0) = \epsilon f(x), \tag{29}$$

where $w(x) = (x + b)(x + 1/b)$, and ϵ is the eigenvalue of $\bar{H}$ ($E = b(\epsilon + b)$). The solution of (29) is straightforward:

$$f(x) = g(x) \int_{x_0}^{x} \frac{bf(0) + f'(0)}{w(x')g(x')} dx', \tag{30}$$

where:

$$g(x) = (x + b)^{-\alpha}(x + 1/b)^{\alpha-1}, \quad \alpha = \frac{\epsilon + b}{b - 1/b} , \quad E = \alpha(b^2 - 1) , \tag{31}$$

is a solution of the homogenous equation and x_0 is to be determined by some boundary condition.

However, since the inhomogeneous term is given by $bf(0) + f'(0)$, there is an additional consistency condition, namely the solution and its derivative, when taken at $x = 0$, must reproduce $f(0)$ and $f'(0)$ again. This leads to the relations

$$\text{either} \quad (bg(0) + g'(0)) = 0, \quad \text{or} \quad \int_{x_0}^{0} dx (x + b)^{\alpha-1}(x + 1/b)^{-\alpha} = 0. \tag{32}$$

The first condition gives $\alpha = 0$, hence it can only lead to zero-energy states. Thus the massive spectrum follows from the second condition (32). Consistency with (22) fixes $x_0 = -1/b$ for $b > 1$ and $x_0 = -b$ for $b < 1$. In either case one should use the analytic continuations of Eq.(30) in order to solve (32).

Once this is done, our solution can be written in terms of the standard hypergeometric function $F \equiv {}_2F_1$ as

$$f(x) = \frac{1}{\alpha} \frac{1}{x + 1/b} F(1, \alpha; 1 + \alpha; \frac{x + b}{x + 1/b}), \quad b < 1, \tag{33}$$

$$f(x) = \frac{1}{1 - \alpha} \frac{1}{x + b} F(1, 1 - \alpha; 2 - \alpha; \frac{x + 1/b}{x + b}), \quad b > 1, \tag{34}$$

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and provides the generating functions for the expansion coefficients $\{c_n\}$ of the arbitrary eigenstate into the $|B_n\rangle$ basis. As one crosscheck examine the $b > 1$ solution at $\alpha = 0$ to find that it indeed generates the second massless state, Eq.(22). On the other hand, similar attempt for the $b < 1$ solution fails – there is no such state in the weak coupling regime.

To summarize, after some trivial change of integration variable, the non-zero-energy levels of the $F = 0$ (and thus by SUSY also of the $F = 1$) sector are given by the roots in α of the following equations:

$$\int_0^{1/\lambda} dx(1-x)^{-1}x^{-\alpha} = 0, \ \ (\lambda > 1) \ ; \ \ \int_0^{\lambda} dx(1-x)^{-1}x^{\alpha-1} = 0, \ \ (\lambda < 1). \qquad (35)$$

Solving these equations indeed reveals a series of discrete zeros, $\alpha_n > (<)0$ for $b > (<)1$ which nicely confirm the eigenvalues $E_n = \alpha_n(\lambda-1)$ computed numerically in the previous section. One immediately checks that, for two values of λ related by $\lambda \to 1/\lambda$, the solutions for α are connected by $\alpha \to 1 - \alpha$, insuring the duality relations (24) among the corresponding eigenvalues.

At this point one can study the flow of the eigenvalues in various situations, e.g. at very weak (and thus by duality also at very strong) coupling. More interesting is the behaviour of the eigenvalues near the critical point at $\lambda = 1$. The Beta-functions in (35) can be related to $F(\alpha, 1, \alpha + 1; \lambda)$ and $F(1 - \alpha, 1, 2 - \alpha; 1/\lambda)$ for $\lambda < 1$ and $\lambda > 1$, respectively. ¿From the known asymptotic behaviour of F (as its last argument approaches 1), we easily get the approximate eigenvalues around $\lambda = 1$ in the form:

$$\begin{aligned} \lambda \to 1^- \ : \ \ & E_n = (-\alpha_n)(1 - \lambda) \,, \\ \lambda \to 1^+ \ : \ \ & E_{n+1} = (1 - \alpha_n)(\lambda - 1) \,, \ n = 0, 1, 2, \dots \,, \\ & \psi(\alpha_n) + \gamma + log(|1 - \lambda|) + O\left(|1 - \lambda|log(|1 - \lambda|)\right) = 0 \,, \end{aligned} \qquad (36)$$

where ψ is the logarithmic derivative of the Γ-function and $\gamma = 0.5772..$ is the Euler-Mascheroni constant. These formulas obey the duality relations (24). They also show the non-analytic way the various levels collapse to zero energy at the critical point. In particular, as $\lambda \to 1^-$, the first eigenvalue approaches zero as $-(1 - \lambda) \ log^{-1}(1 - \lambda)$, i.e. with vanishing first –and infinite second– derivative.

The above formulae also allow a quantitative study of the free energy of the model near the phase transition, which appears to be stronger than in the Gross-Witten model. We have also determined numerically the first few zeroes at generic values of λ, and found perfect agreement with the large cutoff limits of the numerical eigenvalues. The only slightly difficult comparison occurs just around the phase transition where convergence (as one increases the cutoff) undergoes a critical slowdown.

6   Remarks about the $F \geq 2$ sectors

In principle our analysis can be extended in a straightforward way to higher fermion-number sectors. In practice, calculation of the Hamiltonian in those sectors becomes
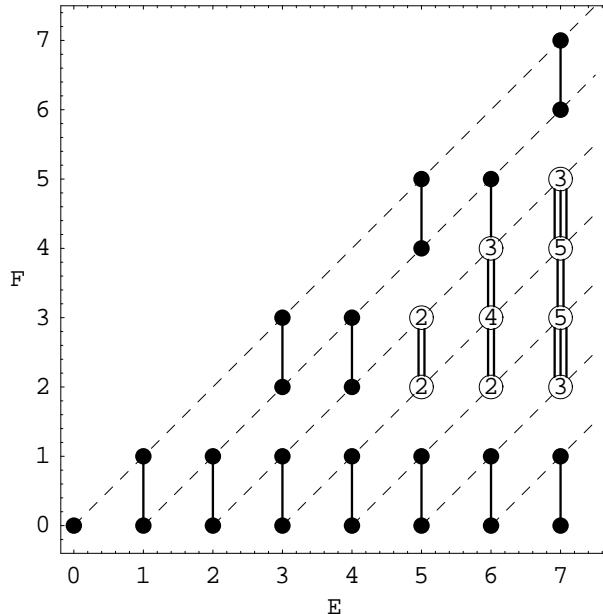
Figure 3: Chew-Frautschi-like plot of the lowest seven levels with their respective degeneracy and supersymmetry partnership at vanishing λ.

quickly cumbersome. There are at least two reasons why it would be worthwhile making such an extension.

Firstly, one would like to check whether the $F = 0, 1$ sectors completely determine the structure of the phase transition at $\lambda = 1$. This would depend on how eigenvalues in the higher-F sectors behave near $\lambda = 1$ and in particular on whether there are discontinuous jumps of Witten's index also in those sectors.

The second reason is that we may expect qualitatively new phenomena to occur when we consider $F \geq 2$ sectors. The fact that eigenstates in the $F = 0$ and $F = 1$ sectors pair nicely without involving, say, $F = 2$ states can be argued on the basis of simply counting the former states at weak coupling. However, when we go to higher F, states are typically highly degenerate at zero coupling. The counting is relatively easy and is summarized in a kind of "Chew-Frautschi" plot in Fig. 3, where non-degenerate states are marked with a full circle while degenerate ones are represented by an open circle showing the degree of degeneracy.

Pairing these states in SUSY doublets (as λ is switched on) turns out to be possible, but non trivial (due to some magic combinatorics! [10]-[12]): it is shown in Fig. 3 via the vertical segments connecting different circles. For instance, the $E = 6$ levels must pair according to the following pattern: the two $F = 2$ states find their SUSY partners in two linear combinations of the four $F = 3$ states. The remaining two $F = 3$ states will match two (linear combinations) of the three $F = 4$ bosons. Finally, the third $F = 4$ boson will pair with the single $F = 5$ fermion. One can also argue that, while most of the above levels have $C/E = (-1)^{F+1}$, two of the $F = 3, E = 6$ levels and one of the $F = 4, E = 6$

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levels have $C/E = (-1)^F$. We are planning to check numerically the low-lying spectra of the $F = 2$ and $F = 3$ sectors in order to see whether, indeed, two of the four $F = 3$ states around $E = 6$ do split from the two $(F = 2, F = 3)$ doublets as we turn on λ. Were this not the case, would signal some higher symmetry underlying the model.

7 Discussion, summary

In this paper we have presented a new way to tackle, both numerically and analytically, planar quantum mechanical problems which hopefully represent the large-N limit of matrix models. Given the ubiquiness of matrix models in theoretical physics (see again [8]), it is hard to overestimate the importance of developing powerful techniques for approaching this kind of questions.

Our method is based on a direct Hamiltonian construction of states and operators that are relevant at lowest genus in a topological expansion of the theory. In principle it should be applicable to any discretization of quantum field theories that allows to compute the planar Hamiltonian in a convenient basis of vectors.

As an illustration of the method we have considered a supersymmetric quantum mechanics model and managed to solve for its spectrum, both numerically and analytically, in two fermionic sectors. Since supersymmetry transformations close within these two sectors, we find, as expected, boson-fermion degeneracy. To our surprise, however, we also find that, at a critical value of the 't Hooft coupling, $\lambda = 1$, the spectrum looses its mass gap and becomes continuous. This conclusion is also confirmed by the radically different dependence of the spectrum on the cutoff at $\lambda = 1$. This dependence is indeed characteristic of the scattering plane-waves [17]. On the other side of the critical value the spectrum has once more a mass gap but there is one more zero-energy bosonic state. In other words the Witten index has jumped by one unit across the phase transition. Furthermore, energy levels on the two sides of the critical point are connected through a non-trivial duality relation. Another amusing property of the model is that, at least within those two sectors, it can be solved analytically.

Besides generalizing the model to more interesting cases, there are two important directions in which the model itself deserves further study:

- Extend calculations to sectors with higher fermion number;

- Understand the situation at finite (though large) N.

We hope to be able to address these issues in a forthcoming paper.

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